Lesson 4-8 Translation Images of Trigonometric Functions

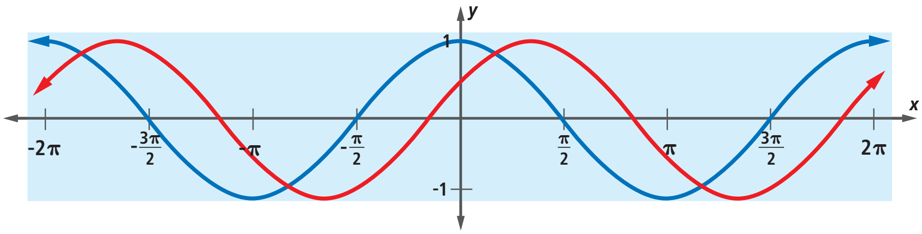
Vocabulary

phase shift

▶ **BIG IDEA** Translations do not affect the period or amplitude of parent trigonometric functions.

Phase Shifts

The graph of $y = \cos x$ is shown in blue along with a graph of a translation image.



Recall from Lesson 3-2 how to translate graphs of functions horizontally and vertically. Horizontal translations of trigonometric functions have a special name: *phase shifts*. This name comes from the study of sound and electricity, where trigonometric functions are often applied. In general, the **phase shift** of a sine wave is the least positive or the greatest negative magnitude of a horizontal translation that maps the graph of $\left(\frac{y}{b}\right) = \cos\left(\frac{x}{a}\right)$ or $\left(\frac{y}{b}\right) = \sin\left(\frac{x}{a}\right)$ onto the wave.

Mental Math

Give the vertex of the parabola described by each equation.

a. $y = -x^2$

b. $y = (x - 5)^2 + 2$

c. $y = 4(x + 50)^2$

d. $y = ax^2 - c$

GUIDED

Example 1

Consider the function h with $h(x) = \sin(x + 60°)$. Identify the phase shift.

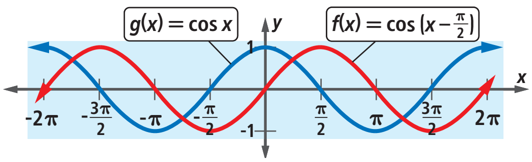
Solution The expression $x + 60°$ is equivalent to $x -$ __?__.
Rewrite the equation as $h(x) = \sin(x - -60°)$. The graph of h is the image of the graph $y = \sin x$ under a horizontal translation of __?__ (left/right). Thus the phase shift is __?__.

People who work with electricity, such as electrical engineers and electricians, use phase shifts. In an alternating current circuit, for example, two waves—the voltage and the current flow—are involved. If these waves coincide, then they are said to be *in phase*. If the current flow lags behind the voltage, then the circuit is *out of phase* and an *inductance* is created. Inductance helps to keep the current flow stable.

Example 2

Maximum inductance in an alternating current occurs when the current flow lags behind the voltage by $\frac{\pi}{2}$. In a situation of maximum inductance, find an equation for the current, and sketch the two waves. Assume that the two waves have the same amplitude and period, and that the voltage is modeled by the equation $y = \cos x$.

Solution Maximum inductance occurs when the current has a phase shift of $\frac{\pi}{2}$. There is no vertical shift. If the equation for the current is of the form $y - k = \cos(x - h)$, then $k = 0$ and $h = \frac{\pi}{2}$. An equation for the current is $y = \cos\left(x - \frac{\pi}{2}\right)$. The waves are graphed at the right.



Examine the graphs carefully. Notice that the graph of $y = \cos\left(x - \frac{\pi}{2}\right)$ seems to coincide with the graph of $y = \sin x$. This relationship gives rise to an identity that is similar to the Complements Theorem.

Theorem (Phase Shift Identity)

For all real numbers x, $\cos\left(x - \frac{\pi}{2}\right) = \sin x$ and $\sin\left(x + \frac{\pi}{2}\right) = \cos x$.

Because the graph of the cosine function is a translation image of the graph of the sine function, these graphs are congruent.

Translation Images of Sine Waves

Example 3

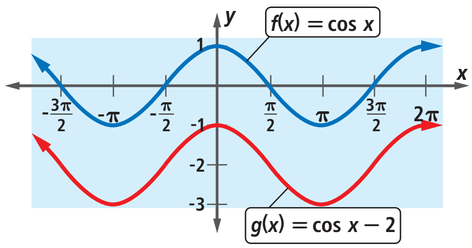
Consider the graph of the function f, where $f(x) = \cos x$.

a. Find an equation for its image g under the translation $(x, y) \rightarrow (x, y - 2)$, and sketch a graph of $y = g(x)$.

b. Find the amplitude and period of the function g.

Solution

a. By the Graph-Translation Theorem, an equation for the image is $y = \cos x - 2$. This results in a vertical shift two units down from the parent function. Graphs of f and g are on the right.



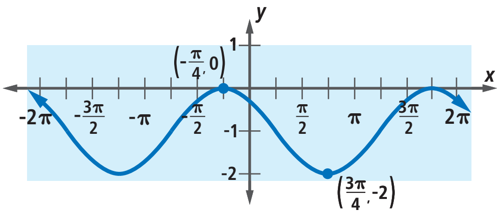
b. The maximum and minimum values of the cosine function are 1 and –1, respectively. So the maximum and minimum values of g are –1 and –3, respectively. Thus, the amplitude of g is $\frac{1}{2}\left|(-1 - -3)\right| = \frac{1}{2} \cdot 2 = 1$, the same as the amplitude of the cosine function. Similarly, the period of g is 2π, the same as the period of the parent function.

In general, if two curves are translation images of each other, then they are congruent. Thus, a translation of a sine or cosine wave preserves both its amplitude and its period.

From an analysis of a graph showing a translation image of any of the parent trigonometric functions, you can determine an equation for a translation image.

Example 4

Find an equation for the translation image of the graph of the cosine function shown below.

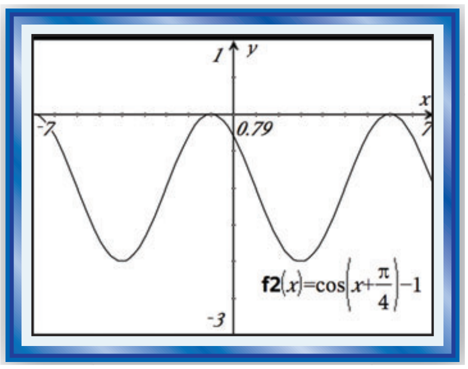


Solution There is both a phase shift and a vertical shift. Since the coordinates of a relative maximum are given, it is convenient to consider $\left(-\frac{\pi}{4}, 0\right)$ as the image of the maximum point (0, 1) of the cosine function. You can consider the point (0, 1) as having been translated $\frac{\pi}{4}$ units to the left and 1 unit down. So the phase shift is $= -\frac{\pi}{4}$.

The vertical shift is 1 unit down. Thus, an equation for this graph is
$y = \cos\left(x + \frac{\pi}{4}\right) - 1$.

The two graphs are congruent, so the amplitude and period remain the same.

Check A graph of $y = \cos\left(x + \frac{\pi}{4}\right) - 1$ shows that the graph corresponds to the given graph.



QY

In Example 4, you could have thought of the graph as a translation image of the sine function. (See Question 7.) In general, when using a sine or cosine function as a model (as you will in Lesson 4-10), the choice of which function is a matter of convenience. Sines and cosines can be used interchangeably if you pay attention to the phase shift.

QY

Identify the phase shift and vertical translation of the graph of $y = \sin x$ so that its image is the graph of $y = \sin\left(x - \frac{\pi}{3}\right) + 4$.

Questions

COVERING THE IDEAS

1. **True or False** The graph of any function and its image under a translation are congruent.

2. Consider the function with equation $y = \cos\left(x + \frac{\pi}{3}\right)$.
   a. Identify the phase shift from $y = \cos x$.
   b. Copy and complete the table at the right, which shows the translation images of the points $(0, 1)$, $\left(\frac{\pi}{2}, 0\right)$, $(\pi, -1)$, $\left(\frac{3\pi}{2}, 0\right)$, and $(2\pi, 1)$.
   c. Use the points found in Part b to help you sketch two cycles of the graph of $y = \cos\left(x + \frac{\pi}{3}\right)$.

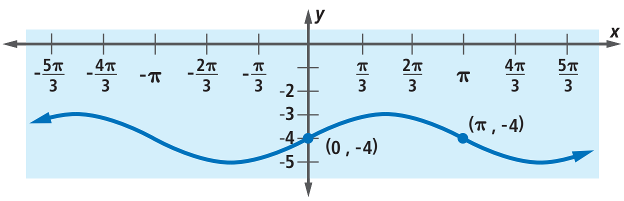
Preimage on $y = \cos x$	Image on $y = \cos\left(x + \frac{\pi}{3}\right)$
$(0, 1)$	?
$\left(\frac{\pi}{2}, 0\right)$	?
$(\pi, -1)$	?
$\left(\frac{3\pi}{2}, 0\right)$	?
$(2\pi, 1)$	?

3. Consider the function with equation $y = \sin\left(x + \frac{3\pi}{4}\right) + 5$
   a. Identify the phase shift from $y = \sin x$.
   b. Identify the vertical shift.

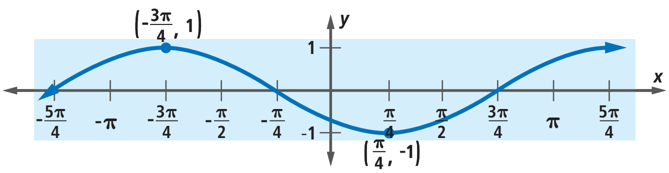
4. Consider the translation $T: (x, y) \rightarrow \left(x + \frac{\pi}{6}, y + 0.5\right)$.
   a. Find an equation for the image of the graph of the sine function under T.
   b. Find the amplitude, the period, and the phase shift of the image.

In 5 and 6, write an equation for the function that is graphed.

5.

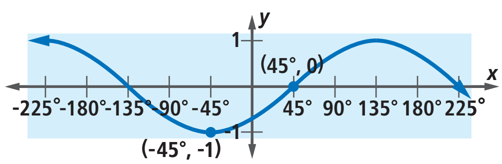


6.



7. Refer to Example 4. Find an equation for the graph, thinking of it as a translation image of the graph of the sine function.

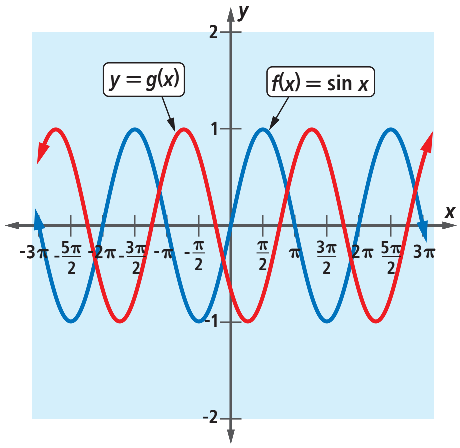
8. Consider the graph at the right.
   a. Write an equation using the sine function to describe the graph.
   b. Write an equation using the cosine function to describe the graph.



9. a. Give an equation for a translation that could be used to map the graph of $f(x) = \sin x$ onto the graph of $g(x) = \cos x$.
   b. Use your answer to Part a to write an expression for $\cos x$ in terms of $\sin x$.

APPLYING THE MATHEMATICS

10. When two waves from the same source travel different distances to reach an object—for example, when radio waves are reflected off of buildings in a city—they may arrive out of phase. Suppose that two such waves are shown in the graph at the right.



   a. Find a formula for $g(x)$.
   b. The signal you would receive is represented by the sum $f + g$. Graph that function.
   c. Find the period, amplitude, and phase shift from the sine function of the function you graphed in Part b.
   d. Identify a value of x for which $f(x) + g(x) = 0$. How can that value be found using just the graphs of f and g?

11. Consider the functions f and g defined by $f(x) = \tan x$ and $g(x) = \tan\left(x - \frac{\pi}{4}\right)$.
   a. Give the period for both f and g.
   b. Describe the transformation that maps the graph of f onto the graph of g.
   c. What is the image of the point $\left(\frac{\pi}{4}, 1\right)$ under this transformation?
   d. Write the equations of the asymptotes of the graph of g on the interval $\frac{\pi}{4} \le x \le \frac{9\pi}{4}$.
   e. Sketch two cycles of the graph of g on the interval $\frac{\pi}{4} \le x \le \frac{9\pi}{4}$.

12. The height h in meters of the tide in a harbor is given by $h = 0.8\cos\left(\frac{\pi}{6}t\right) + 6.5$, where t is the time in hours after high tide.
   a. Calculate h at $t = 0$, $t = 1$, and $t = 2$.
   b. Sketch a graph of this function for $0 \le t \le 24$.
   c. What is the minimum height of the tide during this 24-hour time period?
   d. At what times during the 24 hours after high tide does the minimum height occur?
   e. What is the period of this sine wave and what does the period mean in terms of the tide?

Tides describe the rise and fall in sea level relative to the land due to gravitational pull of the moon and Sun.

REVIEW

In 13 and 14, an equation for a function is given.
a. Find its amplitude. **b. Find its period.** (Lesson 4-7)

13. $y = 4\sin x$

14. $y = 5\cos\left(\frac{x}{2}\right)$

15. Sketch one complete cycle of the graph of $3y = \sin\left(\frac{x}{2}\right)$, and label the zeros of the function shown on the graph. (Lesson 4-7)

In 16 and 17, give the degree equivalent. (Lesson 4-1)

16. 7π

17. $\frac{5\pi}{4}$

18. Let f and g be functions whose equations are $f(x) = x^2 - 3$ and $g(x) = 2x + 1$. Find a formula for $f(g(x))$. (Lesson 3-7)

19. a. Draw $\triangle NOW$, where $N = (2, 6)$, $O = (-1, 4)$, and $W = (1, -4)$.
   b. Draw $\triangle N'O'W'$, the reflection image of $\triangle NOW$ over the y-axis.
   c. If r_y represents reflection over the y-axis, then $r_y : (x, y) \rightarrow$ __?__. (Lesson 3-4)

20. Match each transformation with the *best* description. (Lessons 3-5, 3-4, 3-2)

a. $M(x, y) = (x - 7, y + 5)$ v	(i) reflection over the x-axis
b. $N(x, y) = (y, x)$	(ii) reflection over the line $y = x$
c. $P(x, y) = (x, -y)$	(iii) scale change
d. $Q(x, y) = (0.2x, 20y)$	(iv) size change
e. $V(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right)$	(v) translation

EXPLORATION

21. Noise canceling headphones were developed using the concept of "antiphase."
   a. Find out what antiphase means and how it is used in noise canceling headphones.
   b. If a sound wave is modeled by the equation $y = 12\cos\left(\frac{x}{4}\right)$, write an equation for the antiphase that would cancel the sound.
   c. Find other applications where antiphasing is used.

QY ANSWER

phase shift: $\frac{\pi}{3}$;

vertical translation: 4